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Kilmister's alternative field equations in general relativity: static spherically symmetric solutions

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Abstract. A method of solving the alternative field equations in general relativity proposed by Kilmister is discussed and solutions are found for the case of static spherically symmetric space–times.

1. Introduction

The purpose of this article is to investigate certain alternative field equations in general relativity which have been proposed by Kilmister (1967). We shall attempt to clarify certain points in Kilmister's article, and shall show how solutions may be obtained, confining our attention to static, spherically symmetric space–times. Throughout the article Greek letters are tensor suffixes and Latin letters are ennuple, or tetrad, suffixes. Both types of suffix take the values 1, 2, 3, 4.

Kilmister points out that Einstein's field equations

$$R_{\alpha\beta} = 0 \qquad \text{or} \qquad R_{\alpha\beta} = \lambda g_{\alpha\beta}$$

or, in matter,

$$-\kappa T_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta}(R - 2\lambda)$$

have solutions which are in conflict with Mach's principle. In particular, the Schwarzschild solution describes the field of a single central mass in isolation from the rest of the Universe, whereas, according to Einstein's interpretation of Mach's principle, such a particle should have no mass and no gravitational field. In an attempt to overcome these difficulties, Kilmister reformulates the field equations of general relativity by abandoning the use of covariant differentiation in formulating the geodesic equation in a covariant fashion. Instead he uses an ennuple system, or tetrad, $h_i{}^\alpha$, to define four Lie derivatives, $\underset{i}{\mathscr{L}} v^\alpha$. The ennuple $h_i{}^\alpha$ and its inverse $h^i{}_\alpha$ satisfy

$$h_i{}^\alpha h_{j\alpha} = \eta_{ij} \qquad h_i{}^\alpha h^j{}_\alpha = \delta_i{}^j$$

and

$$h_{i\alpha} h^i{}_\beta = g_{\alpha\beta} \qquad h_i{}^\alpha h^i{}_\beta = \delta^\alpha{}_\beta$$

where η_{ij} is the Minkowski metric tensor $(-1, -1, -1, 1)$. The ennuple can be used to define two affine connections, namely

$$\Delta^\alpha{}_{\beta\gamma} = h_i{}^\alpha h^i{}_{\gamma,\beta} = -h^i{}_\gamma h_i{}^\alpha{}_{,\beta} \tag{1}$$

and

$$\Gamma^\alpha_{\beta\gamma} = h_i{}^\alpha h^i{}_{\beta,\gamma} = -h^i{}_\beta h_i{}^\alpha{}_{,\gamma} \tag{2}$$

so that

$$\Delta^\alpha{}_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}. \tag{3}$$

We denote covariant differentiations with respect to $\Delta^\alpha{}_{\beta\gamma}$ and $\Gamma^\alpha_{\beta\gamma}$ by a single line | and a double line || respectively. Now the Lie derivatives $\underset{i}{\mathscr{L}} v^\alpha$ are given by

$$\underset{i}{\mathscr{L}} v^\alpha = (h_i{}^\beta v^\alpha{}_{,\beta} - h_i{}^\alpha{}_{,\beta} v^\beta)$$

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so that

$$
\begin{aligned}
v^i \underset{i}{\mathscr{L}} v^\alpha &= v^\gamma h^i{}_\gamma (h_i{}^\beta v^\alpha{}_{,\beta} - h_i{}^\alpha{}_{,\beta} v^\beta) \\
&= v^\beta v^\alpha{}_{,\beta} + v^\beta \Delta^\alpha{}_{\beta\gamma} v^\gamma \\
&= v^\beta v^\alpha{}_{,\beta} + v^\beta \Delta^\alpha{}_{\gamma\beta} v^\gamma \qquad (4)\\
&= v^\beta v^\alpha{}_{|\beta} \qquad (5)\\
&= v^\beta v^\alpha{}_{;\beta} + C^\alpha{}_{\beta\gamma} v^\beta v^\gamma
\end{aligned}
$$

where

$$C^\alpha{}_{\beta\gamma} = \Delta^\alpha{}_{\beta\gamma} - \left\{ {\alpha \atop \beta\gamma} \right\} \tag{6}$$

and the semicolon denotes covariant differentiation with respect to the Christoffel bracket connection $\left\{ {\alpha \atop \beta\gamma} \right\}$. Hence the equation of a geodesic is

$$v^i \underset{i}{\mathscr{L}} v^\alpha = C^\alpha{}_{\beta\gamma} v^\beta v^\gamma.$$

Note that from equations (3) and (4) it follows that (5) can be written in the form

$$v^i \underset{i}{\mathscr{L}} v^\alpha = v^\beta v^\alpha{}_{\|\beta}.$$

The affine connection $\Gamma^\alpha_{\beta\gamma}$, defined in (2), is such that the covariant derivatives of the $h_i{}^\alpha$ are zero with respect to it, since

$$
\begin{aligned}
h_i{}^\alpha{}_{\|\beta} &= h_i{}^\alpha{}_{,\beta} + \Gamma^\alpha_{\gamma\beta} h_i{}^\gamma \\
&= h_i{}^\alpha{}_{,\beta} - h^j{}_\gamma h_j{}^\alpha{}_{,\beta} h_i{}^\gamma \\
&= h_i{}^\alpha{}_{,\beta} - \delta_i{}^j h_j{}^\alpha{}_{,\beta} \\
&= 0.
\end{aligned}
$$

It follows that the curvature tensor of $\Gamma^\alpha_{\beta\gamma}$ is zero. Now the equation $C^\alpha{}_{\beta\gamma} = 0$ implies, from (6), that

$$\Delta^\alpha{}_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\}$$

and so from (1) and (2)

$$\Delta^\alpha{}_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \left\{ {\alpha \atop \beta\gamma} \right\}.$$

Hence $\left\{ {\alpha \atop \beta\gamma} \right\}$ has zero-curvature tensor and the space is flat. Kilmister then states the converse, i.e. that the presence of a gravitational field is manifested by non-vanishing $C^\alpha{}_{\beta\gamma}$. We shall clarify this statement in § 3.

Kilmister takes the view that the gravitational field is described by all sixteen ennuple components, so that it is the ennuple, rather than the metric tensor, which has the primary physical significance. Accordingly, sixteen field equations are required and Kilmister suggests

$$C^\alpha{}_{\beta\gamma;\alpha} = 0 \qquad \text{or} \qquad C^\alpha{}_{\beta\gamma|\alpha} = 0.$$

Kilmister calculates that

$$C^\alpha{}_{\beta\gamma;\alpha} = R_{\beta\gamma} - K_{\beta\gamma} + C^\alpha{}_{\beta\alpha|\gamma} + C^\alpha{}_{\beta\delta} C^\delta{}_{\alpha\gamma} \tag{7}$$

and

$$C^\alpha{}_{\beta\gamma|\alpha} = R_{\beta\gamma} - K_{\beta\gamma} + C^\alpha{}_{\beta\alpha;\gamma} - C^\alpha{}_{\beta\delta} C^\delta{}_{\gamma\alpha} \tag{8}$$

where $K_{\beta\gamma}$ is the contracted curvature tensor formed from $\Delta^\alpha{}_{\beta\gamma}$. By considering the third term on the right-hand side of each equation, Kilmister selects the first equation as the more suitable. However, as we shall show in § 2, $C^{\alpha}{}_{\beta\alpha}$ is zero, so the choice between the two equations must depend only upon the solutions that they permit.

The basic mathematics described in this and in the subsequent section can be found in greater detail in the works of Cartan (1951) and Schouten (1954).

2. The tensor $C^{\alpha}{}_{\beta\gamma}$ and possible field equations

From equations (1) and (6) we have that

$$C^{\alpha}{}_{\beta\gamma} = h_i{}^{\alpha} h^i{}_{\gamma,\beta} - \left\{{}^{\alpha}_{\beta\gamma}\right\}$$

$$= h_i{}^{\alpha}(h^i{}_{\gamma,\beta} - \left\{{}^{\delta}_{\gamma\beta}\right\} h^i{}_{\delta})$$

i.e.

$$C^{\alpha}{}_{\beta\gamma} = h_i{}^{\alpha} h^i{}_{\gamma;\beta}. \tag{9}$$

If follows that

$$h_i{}^{\alpha} h_j{}^{\gamma} h_k{}^{\beta} C_{\alpha\beta\gamma} = h_{i\gamma;\beta} h_j{}^{\gamma} h_k{}^{\beta} = \gamma_{ijk}$$

where

$$C_{\alpha\beta\gamma} = g_{\alpha\delta} C^{\delta}{}_{\beta\gamma}$$

and γ_{ijk} are the Ricci rotation coefficients (Eisenhart 1926). Since the γ_{ijk} are antisymmetric in the suffixes i and j, it follows that $C_{\alpha\beta\gamma}$ is antisymmetric in α and γ and hence

$$C^{\alpha}{}_{\beta\alpha} = 0$$

so that the reasons for Kilmister's choice between his two sets of suggested field equations no longer hold. The equations (7) and (8) now read

$$C^{\alpha}{}_{\beta\gamma;\alpha} = R_{\beta\gamma} - K_{\beta\gamma} + C^{\alpha}{}_{\beta\delta} C^{\delta}{}_{\alpha\gamma} \tag{10}$$

$$C^{\alpha}{}_{\beta\gamma|\alpha} = R_{\beta\gamma} - K_{\beta\gamma} - C^{\alpha}{}_{\beta\delta} C^{\delta}{}_{\gamma\alpha}. \tag{11}$$

The field formed by equating expressions (10) and (11) to zero are of the same type as Einstein's field equations, i.e. they are covariant second-order differential equations for the metric tensor, and hence of the ennuple, and linear in the second-order terms. The question now arises: what other equations of this type can be formed from the $C^{\alpha}{}_{\beta\gamma}$? One obvious equation is

$$C^{\alpha}{}_{\beta\gamma\|\alpha} = R_{\beta\gamma} - K_{\beta\gamma} + C^{\alpha}{}_{\beta\gamma} C_{\alpha} - C^{\alpha}{}_{\delta\beta} C^{\delta}{}_{\alpha\gamma} = 0 \tag{12}$$

where $C_{\alpha} = C^{\beta}{}_{\beta\alpha}$, and is not to be confused with Kilmister's $C_{\alpha} = C^{\beta}{}_{\alpha\beta}$ which, as we have seen, is zero. Another equation suggests itself when we note that

$$g_{\alpha\beta;\gamma} = 0 \qquad \text{and} \qquad g_{\alpha\beta\|\gamma} = 0$$

but

$$g_{\alpha\beta|\gamma} \neq 0$$

so that we can consider the equation

$$g^{\alpha\delta} C_{\delta\beta\gamma|\alpha} = R_{\beta\gamma} - K_{\beta\gamma} + C^{\alpha}{}_{\beta\gamma} C_{\alpha} - C^{\alpha}{}_{\beta\delta} C^{\delta}{}_{\gamma\alpha} = 0. \tag{13}$$

Note that $g^{\alpha\delta} C_{\delta\beta\gamma|\alpha} = -g^{\alpha\delta} C_{\gamma\beta\delta|\alpha}$, whilst $g^{\alpha\beta} C_{\delta\beta\gamma|\alpha}$ has six components only, so (13) is the only possible covariant equation of its form. Note also that the mixed and contravariant forms of (13) will have additional terms due to the non-vanishing covariant derivative of $g_{\alpha\beta}$ with respect to $\Delta^{\alpha}{}_{\beta\gamma}$.

Further field equations of this type can be formed by contracting the tensor $C^{\alpha}{}_{\beta\gamma}$ before differentiating, for example

$$C^{\gamma}{}_{\gamma\alpha;\beta} \equiv C_{\alpha;\beta} = 0 \tag{14}$$

is an equation of the required type with sixteen independent components. Other such equations are

$$C_{\alpha|\beta} = C_{\alpha;\beta} - C^{\gamma}{}_{\alpha\beta}C_{\gamma} = 0 \quad (15)$$

$$C_{\alpha\|\beta} = C_{\alpha;\beta} - C^{\gamma}{}_{\beta\alpha}C_{\gamma} = 0 \quad (16)$$

and

$$g^{\alpha\delta}C_{\delta\alpha\beta|\gamma} = C_{\beta;\gamma} - C^{\delta}{}_{\beta\gamma}C_{\delta} - C^{\alpha\delta}{}_{\gamma}C_{\delta\alpha\beta} - C^{\delta\alpha}{}_{\gamma}C_{\delta\alpha\beta} = 0. \quad (17)$$

There appears to be no reason, other than the solutions they admit, why any one of the equations (10)–(17) should be preferred to the others. In this article we shall consider only Kilmister's choice $C^{\alpha}{}_{\beta\gamma;\alpha} = 0$.

3. The ennuple in flat space–time

In a flat space–time with Cartesian coordinates, i.e. with metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \equiv (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad (18)$$

the ennuple $h_i{}^{\alpha}$ satisfies

$$h_i{}^{\alpha}h_{j\alpha} = \eta_{ij} \quad \text{and} \quad h^i{}_{\alpha}h_{i\beta} = \eta_{\alpha\beta}. \quad (19)$$

The general solution of equations (19) contains six arbitrary parameters corresponding to the six parameters of the group of Lorentz rotations. If we write the components of the four 4-vectors $h_i{}^{\alpha}$ as a 4×4 matrix, the rows representing each vector, then the matrix $h_i{}^{\alpha}$ satisfying equations (18) is, in general, any product of the six matrices $H_{23}(\alpha_1)$, $H_{31}(\alpha_2)$, $H_{12}(\alpha_3)$, $H_{14}(\beta_1)$, $H_{24}(\beta_2)$, $H_{34}(\beta_3)$ given by

$$H_{23}(\alpha_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 & 0 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for spatial rotations about the x axis, and two similar matrices $H_{31}(\alpha_2)$, $H_{12}(\alpha_3)$, for spatial rotations about the y and z axes, and also

$$H_{14}(\beta_1) = \begin{bmatrix} \cosh\beta_1 & 0 & 0 & \sinh\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta_1 & 0 & 0 & \cosh\beta_1 \end{bmatrix}$$

for rotations in the (x, t) plane, and two similar matrices $H_{24}(\beta_2)$, $H_{34}(\beta_3)$, for rotations in the (y, t) and (z, t) planes.

We shall call the three parameters α_1, α_2, α_3 the 'circular parameters' and the three parameters β_1, β_2, β_3 the 'hyperbolic parameters'. These six parameters are, in general, functions of the four coordinates x^{α}. The product of the six matrices, in any order, will be denoted by $\mathscr{H}$. The form of $\mathscr{H}$ will depend on the order of multiplication, but whatever that order, $\mathscr{H}$ will always satisfy (19), i.e. in matrix form

$$\mathscr{H}\mathscr{E}\mathscr{H}^{\mathrm{T}} = \mathscr{E} \quad (20)$$

where $\mathscr{E}$ is the matrix

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we use coordinates other than Cartesians, (19) and (20) must be replaced by

$$h_i^{\alpha}h_{j\alpha} = h_i^{\alpha}g_{\alpha\beta}h_j^{\beta} = \eta_{ij} \qquad \text{and} \qquad h^i{}_{\alpha}h_{i\beta} = g_{\alpha\beta} \tag{21}$$

and

$$\mathscr{H}\,\mathscr{F}\,\mathscr{G}\,\mathscr{F}^{\mathrm{T}}\mathscr{H}^{\mathrm{T}} = \mathscr{E} \tag{22}$$

respectively, where $\mathscr{G}$ is the matrix of the components $g_{\alpha\beta}$ of the metric tensor, and $\mathscr{F}$ is a matrix satisfying

$$\mathscr{F}\,\mathscr{G}\,\mathscr{F}^{\mathrm{T}} = \mathscr{E}. \tag{23}$$

The matrix representation of the ennuple components h_i^{α} will then be

$$h_i^{\alpha} = \mathscr{H}\,\mathscr{F}. \tag{24}$$

For example, consider the flat space–time with spherical polar coordinates, i.e.

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{25}$$

In this case $\mathscr{F}$ is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1}{r} & 0 & 0 \\ 0 & 0 & \dfrac{1}{r\sin\theta} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{26}$$

Consider now the tensor $C^{\alpha}{}_{\beta\gamma}$ in a flat space–time. Kilmister showed that the vanishing of $C^{\alpha}{}_{\beta\gamma}$ implied that space–time was flat, but what of the converse result? If we use Cartesian coordinates, i.e. a metric of the form (18), then, since the $\left\{{\alpha \atop \beta\gamma}\right\}$ are all zero, we have

$$C^{\alpha}{}_{\beta\gamma} = h_i^{\alpha}h^i{}_{\gamma,\beta}$$

so that $C^{\alpha}{}_{\beta\gamma} = 0$ if, and only if, $h^i{}_{\gamma,\beta} = 0$, i.e. if the ennuple components are constants. Since the h_i^{α} are unit vectors it follows that, by a constant four-dimensional rotation, the ennuple components can be reduced to the form

$$h_i^{\alpha} = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}. \tag{27}$$

This ennuple corresponds to the general ennuple, represented by $\mathscr{H}$, with all the six parameters α_1, α_2, α_3, β_1, β_2, β_3 set equal to zero.

The dependence of the vanishing of $C^{\alpha}{}_{\beta\gamma}$ on the choice of ennuple can be illustrated by considering the two-dimensional flat space

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \equiv (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2.$$

The most general ennuple depends on one parameter $\psi = \psi(x, y)$ only and can be written in the form

$$h_i^{\alpha} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}. \tag{28}$$

The non-zero components of $C^{\alpha}{}_{\beta\gamma}$ are

$$C^2_{11} = -C^1{}_{12} = \psi_{,1}$$

$$C^2_{21} = -C^1{}_{22} = \psi_{,2}.$$

Hence, $C^{\alpha}{}_{\beta\gamma} = 0$ if, and only if, ψ is a constant. In this case the ennuple can be taken, apart from constant rotations, to be of the form

$$h_i{}^{\alpha} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{29}$$

If we introduce polar coordinates so that the metric of the 2-space is

$$ds^2 = dr^2 + r^2\, d\theta^2$$

then the general ennuple is of the form

$$h_i{}^{\alpha} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{r} \end{bmatrix} = \begin{bmatrix} \cos\psi & -\dfrac{1}{r}\sin\psi \\ \sin\psi & \dfrac{1}{r}\cos\psi \end{bmatrix}. \tag{30}$$

With this ennuple the components of $C^{\alpha}{}_{\beta\gamma}$ are

$$C^1{}_{12} = -r\psi_{,1}$$

$$C^2{}_{11} = \frac{1}{r}\psi_{,1}$$

$$C^2{}_{21} = \frac{1}{r}(\psi_{,2}-1)$$

$$C^1{}_{22} = -r(\psi_{,2}-1)$$

where the comma followed by 1 or 2 now indicates partial differentiation with respect to r, θ respectively. Thus $C^{\alpha}{}_{\beta\gamma} = 0$ if, and only if, $\psi = \theta$ (apart from a constant rotation), so that the required ennuple is

$$h_i{}^{\alpha} = \begin{bmatrix} \cos\theta & -\dfrac{1}{r}\sin\theta \\ \sin\theta & \dfrac{1}{r}\cos\theta \end{bmatrix} \tag{31}$$

which is precisely the ennuple obtained from (29) by the coordinate transformation $(x, y) \to (r, \theta)$, as expected.

Note that if we had chosen the simplest ennuple for this coordinate system, i.e.

$$h_i{}^{\alpha} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{r} \end{bmatrix}$$

which corresponds to (30) with $\psi = 0$, then not only would $C^{\alpha}{}_{\beta\gamma} \neq 0$ but also $C^{\alpha}{}_{\beta\gamma;\alpha} \neq 0$. Thus, despite having a flat space, the vacuum-field equations are not satisfied if the ennuple is assumed to be too simple. In this case we have

$$C^2{}_{21} = -\frac{1}{r} \qquad C^1{}_{22} = r$$

$$C^\alpha{}_{11;\alpha} = \frac{1}{r^2} \qquad C^\alpha{}_{22;\alpha} = -1.$$

Returning to Minkowski space–time, consider the metric (25) expressed in spherical polar coordinates. Under the transformation $(x, y, z) \to (r, \theta, \phi)$ the ennuple (27) becomes

$$h_i{}^\alpha = \begin{bmatrix} \sin\theta\cos\phi & \frac{1}{r}\cos\theta\cos\phi & -\frac{1}{r}\frac{\sin\phi}{\sin\theta} & 0 \\ \sin\theta\sin\phi & \frac{1}{r}\cos\theta\sin\phi & \frac{1}{r}\frac{\cos\phi}{\sin\theta} & 0 \\ \cos\theta & -\frac{1}{r}\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{32}$$

Comparing this with (24) and noting that $\mathscr{F}$ is given by (26) we have

$$\mathscr{H} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi & 0 \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{33}$$

This corresponds to the general product matrix of the six Lorentz rotation matrices with $\beta_1 = \beta_2 = \beta_3 = 0$ and $\alpha_1 = \frac{1}{2}\pi$, $\alpha_2 = \theta - \frac{1}{2}\pi$, $\alpha_3 = -\phi$, so that the ennuple now has three non-zero parameters. These values for the circular parameters are obtained by considering the product of the circular matrices in the order $H_{12}H_{31}H_{23}$; other orders of multiplication will lead to different values for α_1, α_2, α_3. The ennuple (32) is, apart from constant rotations, the only ennuple leading to $C^\alpha{}_{\beta\gamma} = 0$ for the space–time (25).

If we write the Minkowski space–time in terms of cylindrical polar coordinates in the form

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - dz^2$$

then under the transformation $(x, y, z) \to (r, \theta, z)$ the ennuple (27) becomes

$$h_i{}^\alpha = \begin{bmatrix} \cos\theta & -\frac{1}{r}\sin\theta & 0 & 0 \\ \sin\theta & \frac{1}{r}\cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the corresponding product matrix $\mathscr{H}$ has only one non-zero parameter $\alpha_3 = -\theta$.

Consider now the space–time

$$ds^2 = d\tau^2 - d\xi^2 - u^2(d\eta^2 + d\zeta^2)$$

where $u = \tau - \xi$. This is a flat space–time and is a special case of the metric used by Bondi *et al.* (1959). This metric can be obtained from (18) by the transformation

$$\begin{aligned} x &= \xi + \tfrac{1}{2}(\tau-\xi)(\eta^2+\zeta^2) \\ y &= \eta(\tau-\xi) \\ z &= \zeta(\tau-\xi) \\ t &= \tau + \tfrac{1}{2}(\tau-\xi)(\eta^2+\zeta^2). \end{aligned}$$

Under this transformation the ennuple (27) becomes

$$h_i^{\ \alpha} = \begin{bmatrix} -\tfrac{1}{2}(\eta^2+\zeta^2) & 1-\tfrac{1}{2}(\eta^2+\zeta^2) & \dfrac{1}{u}\eta & \dfrac{1}{u}\zeta \\ -\eta & -\eta & \dfrac{1}{u} & 0 \\ -\zeta & -\zeta & 0 & \dfrac{1}{u} \\ 1+\tfrac{1}{2}(\eta^2+\zeta^2) & \tfrac{1}{2}(\eta^2+\zeta^2) & -\dfrac{1}{u}\eta & -\dfrac{1}{u}\zeta \end{bmatrix}$$

and the corresponding product matrix $\mathscr{H}$ has six non-zero parameters given by

$$\begin{aligned} \sin\alpha_1 &= 2\eta\{4(1-\zeta^2)+(\eta^2+\zeta^2)^2\}^{-1/2} \\ \sin\alpha_2 &= 2\zeta\{4+(\eta^2+\zeta^2)^2\}^{-1/2} \\ \sin\alpha_3 &= 2\eta\zeta(\eta^2+\zeta^2)[\{4(1-\zeta^2)+(\eta^2+\zeta^2)^2\}\{4(1+\eta^2)+(\eta^2+\zeta^2)^2\}]^{-1/2} \\ \sinh\beta_1 &= \tfrac{1}{2}(\eta^2+\zeta^2) \\ \sinh\beta_2 &= -2\eta\{4+(\eta^2+\zeta^2)^2\}^{-1/2} \\ \sinh\beta_3 &= -2\zeta\{4(1+\eta^2)+(\eta^2+\zeta^2)^2\}^{-1/2}. \end{aligned}$$

These values are found by taking the order of multiplication to be

$$\mathscr{H} = H_{12}H_{23}H_{31}H_{34}H_{24}H_{14};$$

other orders of multiplication will lead to different values for the six parameters.

Summarizing, we have shown that in a flat space–time the tensor $C^{\alpha}{}_{\beta\gamma}$ will vanish if, and only if, we use the ennuple (27) in Cartesian coordinates (ignoring constant rotations), or the corresponding transformed ennuple in other coordinate systems. We shall call this appropriate ennuple the 'proper ennuple' for the coordinate system used. All proper ennuples revert to the form (27) on transforming to Cartesian coordinates. The number of non-zero parameters associated with a proper ennuple depends on the coordinate system: for Cartesian coordinates the number is zero; for other coordinate systems the number n of parameters is such that $0 < n \leqslant 6$.

4. The ennuple in curved space–time

In a curved space–time we no longer have the condition $C^{\alpha}{}_{\beta\gamma} = 0$ to dictate the choice of ennuple. Instead, the field equations $C^{\alpha}{}_{\beta\gamma;\alpha} = 0$ provide the sixteen equations for the ten components of the metric tensor $g_{\alpha\beta}$ and the six parameters appearing in the $h_i^{\ \alpha}$. In general, these six parameters are functions of all four coordinates and, as a result, the solution of the field equations presents an intractable problem. In order to be able to find solutions we must place some restrictions on the parameters, but we must be careful not to be too restrictive or we may obtain equations that do not have solutions. For example, in the previous section, when considering polar coordinates in a two-dimensional flat space, we saw that by choosing the very simple ennuple corresponding to the parameter $\psi = 0$, the field equations $C^{\alpha}{}_{\beta\gamma;\alpha} = 0$ were not satisfied. Similarly, in a curved space we must not oversimplify the ennuple in our desire to obtain equations which we can reasonably hope to solve.

We wish to solve the field equations for a static spherically symmetric space–time whose metric can be written in the form

$$\mathrm{d}s^2 = A(r)\,\mathrm{d}t^2 - B(r)\,\mathrm{d}r^2 - C(r)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$

If $C(r)$ is not a constant the metric can be written in the form

$$\mathrm{d}s^2 = \mathrm{e}^{2\mu}\,\mathrm{d}t^2 - \mathrm{e}^{2\nu}\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{34}$$

whereas if $C(r)$ is a constant, $C(r) = a$, then the metric is

$$\mathrm{d}s^2 = \mathrm{e}^{2\mu}\,\mathrm{d}t^2 - \mathrm{e}^{2\nu}\,\mathrm{d}r^2 - a^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{35}$$

Consider first space–times with metrics of the form (34). The general ennuple is given by

$$h_i{}^{\alpha} = \mathscr{H}\,\mathscr{F}$$

where $\mathscr{H}$ is the product of the Lorentz matrices and $\mathscr{F}$ is given by

$$\mathscr{F} = \begin{bmatrix} \mathrm{e}^{-\nu} & & & 0 \\ & \dfrac{1}{r} & & \\ & & \dfrac{1}{r\sin\theta} & \\ 0 & & & \mathrm{e}^{-\mu} \end{bmatrix}. \tag{36}$$

It is tempting to use the simple ennuple given by taking $\mathscr{H}$ as the unit matrix, i.e. by taking all the six parameters to be zero, but, as in the two-dimensional polar coordinate case, it is easily shown that no solution of the field equations exists for this ennuple. The discussion of the previous section suggests that we should look at the proper ennuple (32) for a Minkowski space–time with spherical polar coordinates and adapt this ennuple to the space–time given by (34). In other words, we use the ennuple

$$h_i{}^{\alpha} = \mathscr{H}\,\mathscr{F}$$

where $\mathscr{H}$ is given by (33) and $\mathscr{F}$ by (36). Throughout the rest of this paper, $\mathscr{H}$ given by (33) will be denoted by Θ. As stated earlier, Θ is the product of the three circular Lorentz matrices with the product taken in any order, the identification of the parameters depending on the order of multiplication. This matrix Θ is fundamentally related to the use of spherical polar coordinates in Minkowski space.

We now ask whether it is possible to make this ennuple slightly more general. In particular we would like to consider the possibility of introducing the hyperbolic parameters into the ennuple. These hyperbolic parameters are, in general, functions of all four coordinates, but here we shall take them to be functions of r only. Our reasons for making this restriction are twofold. Firstly, to assume dependence on all four coordinates would lead to enormous difficulties in the calculations, so our assumption is essentially for expediency. Secondly, Plebanski (1962) has drawn attention to what he calls the "unconventional" view in which the ennuple components represent something physical rather than being simply a useful mathematical tool. This view was originally expressed by Einstein (1928). If we adopt it here it is reasonable to expect that in a physical situation, which is static and spherically symmetric, the ennuple will be a function of r only, apart from the contribution due to the matrix Θ, which is essentially a manifestation of the coordinate system. Hence we shall take the ennuple to be of the restricted form

$$h_i^{\alpha} = \Theta\Phi\mathscr{F} \tag{37}$$

where Φ is a product of three matrices H_{14}, H_{24}, H_{34} and the parameters β_1, β_2, β_3 are assumed to be functions of r only. Now in this product the three hyperbolic matrices can be taken in any order and in (37) Φ can come before Θ or it can be split, with some of the three hyperbolic matrices pre-multiplying Θ and some post-multiplying Θ. For the purpose of solving the field equations we shall take the ennuple to be given by (37) with Φ given by

$$\Phi = H_{34}(\beta_3)H_{24}(\beta_2)H_{14}(\beta_1) \tag{38}$$

in that order. Later we shall discuss the results of taking different orders of multiplication.

The four ennuple vectors are thus given as the rows of the matrix product

$$h_i^{\alpha} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi & 0 \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \operatorname{ch}\beta_1 & 0 & 0 & \operatorname{sh}\beta_1 \\ \operatorname{sh}\beta_1\operatorname{sh}\beta_2 & \operatorname{ch}\beta_2 & 0 & \operatorname{ch}\beta_1\operatorname{sh}\beta_2 \\ \operatorname{sh}\beta_1\operatorname{ch}\beta_2\operatorname{sh}\beta_3 & \operatorname{sh}\beta_2\operatorname{sh}\beta_3 & \operatorname{ch}\beta_3 & \operatorname{ch}\beta_1\operatorname{ch}\beta_2\operatorname{sh}\beta_3 \\ \operatorname{sh}\beta_1\operatorname{ch}\beta_2\operatorname{ch}\beta_3 & \operatorname{sh}\beta_2\operatorname{ch}\beta_3 & \operatorname{sh}\beta_3 & \operatorname{ch}\beta_1\operatorname{ch}\beta_2\operatorname{ch}\beta_3 \end{bmatrix}$$

$$\times \begin{bmatrix} e^{-\nu} & & & \\ & \frac{1}{r} & & 0 \\ & & \frac{1}{r\sin\theta} & \\ 0 & & & e^{-\mu} \end{bmatrix} \tag{39}$$

where we have written sh, ch for sinh, cosh, and $\beta_1 = \beta_1(r)$, $\beta_2 = \beta_2(r)$, $\beta_3 = \beta_3(r)$.

By exactly similar reasoning to that given above, the ennuple for space–time of the form (35) is given by

$$h_i^{\alpha} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi & 0 \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \operatorname{ch}\beta_1 & 0 & 0 & \operatorname{sh}\beta_1 \\ \operatorname{sh}\beta_1 \operatorname{sh}\beta_2 & \operatorname{ch}\beta_2 & 0 & \operatorname{ch}\beta_1 \operatorname{sh}\beta_2 \\ \operatorname{sh}\beta_1 \operatorname{ch}\beta_2 \operatorname{sh}\beta_3 & \operatorname{sh}\beta_2 \operatorname{sh}\beta_3 & \operatorname{ch}\beta_3 & \operatorname{ch}\beta_1 \operatorname{ch}\beta_2 \operatorname{sh}\beta_3 \\ \operatorname{sh}\beta_1 \operatorname{ch}\beta_2 \operatorname{ch}\beta_3 & \operatorname{sh}\beta_2 \operatorname{ch}\beta_3 & \operatorname{sh}\beta_3 & \operatorname{ch}\beta_1 \operatorname{ch}\beta_2 \operatorname{ch}\beta_3 \end{bmatrix}$$

$$\times \begin{bmatrix} e^{-\nu} & & & 0 \\ & \frac{1}{a} & & \\ & & \frac{1}{a\sin\theta} & \\ 0 & & & e^{-\mu} \end{bmatrix} \tag{40}$$

where, as before, $\beta_1 = \beta_1(r)$, $\beta_2 = \beta_2(r)$, $\beta_3 = \beta_3(r)$.

It must be emphasized that the main purpose of the restrictions imposed on the ennuple is to reduce the field equations to a set of differential equations which we have some reasonable hope of solving. If the resulting equations have no solution, this may be due to the restrictions on the ennuple being too strong or of the wrong type. On the other hand, if we can solve the equations we cannot be sure that we have found all the static spherically symmetric solutions of the field equations, since a less restricted ennuple may lead to further solutions.

5. Solutions of the field equations

With the ennuple given by (39), the non-zero components of the tensor $C^{\alpha}{}_{\beta\gamma}$ are found to be

$$C^1{}_{12},\ C^1{}_{22},\ C^1{}_{13},\ C^1{}_{23},\ C^1{}_{33},\ C^1{}_{14},\ C^1{}_{24},\ C^1{}_{34},\ C^1{}_{44}$$
$$C^2{}_{11},\ C^2{}_{21},\ C^2{}_{13},\ C^2{}_{23},\ C^2{}_{33},\ C^2{}_{14},\ C^2{}_{24},\ C^2{}_{34}$$
$$C^3{}_{11},\ C^3{}_{21},\ C^3{}_{31},\ C^3{}_{12},\ C^3{}_{22},\ C^3{}_{32},\ C^3{}_{14},\ C^3{}_{34}$$
$$C^4{}_{11},\ C^4{}_{21},\ C^4{}_{31},\ C^4{}_{41},\ C^4{}_{12},\ C^4{}_{22},\ C^4{}_{32},\ C^4{}_{13},\ C^4{}_{33}.$$

The sixteen field equations $C^{\alpha}{}_{\beta\gamma;\alpha} = 0$ can be computed from the values of $C^{\alpha}{}_{\beta\gamma}$. These field equations are complicated, but an immediate simplification is found if we first consider the (2, 2) equation, which is of the form

$$f(r) - \cot\theta\{\operatorname{sh}\beta_1 \operatorname{sh}\beta_2 \operatorname{ch}\beta_3 + \cot\theta(\operatorname{ch}\beta_2 \operatorname{ch}\beta_3 - 1)\} = 0$$

where $f(r)$ is a function of r only. Since the coefficient of $\cot^2\theta$ must be zero, we have

$$\beta_2 = \beta_3 = 0.$$

Putting β_2, β_3 equal to zero we find that the only field equations which are not identically zero are

$$C^{\alpha}{}_{11;\alpha} \equiv -\frac{2}{r^2}(e^{\nu}\operatorname{ch}\beta_1 - 1) + \mu'^2 = 0 \tag{41}$$

$$C^{\alpha}{}_{22;\alpha} \equiv -(\nu' - \mu')r\,e^{-2\nu} - \mu' r\,e^{-\nu}\operatorname{ch}\beta_1 - \beta_1' r\,e^{-\nu}\operatorname{sh}\beta_1 = 0 \tag{42}$$

$$C^{\alpha}{}_{33;\alpha} \equiv C^{\alpha}{}_{22;\alpha}\sin^2\theta$$

$$C^{\alpha}{}_{44;\alpha} \equiv \left(-\mu'' + \mu'\nu' - \frac{2\mu'}{r}\right)e^{2\mu - 2\nu} = 0 \tag{43}$$

$$C^{\alpha}{}_{41;\alpha} \equiv -\mu'\beta_1' \, e^{\mu-\nu} = 0 \tag{44}$$

$$C^{\alpha}{}_{14;\alpha} \equiv -\beta_1'' \, e^{\mu-\nu} + \beta_1'\nu' \, e^{\mu-\nu} - \frac{2}{r}\beta_1' \, e^{\mu-\nu} + \frac{2}{r^2} \operatorname{sh}\beta_1 \, e^{\mu} = 0. \tag{45}$$

From (44), either $\mu' = 0$ or $\beta_1' = 0$, and it is easily seen that $\beta_1' = 0$ leads to a contradiction, so we must have $\mu' = 0$ and we can take $\mu = 0$ without loss of generality. Putting $\mu = 0$ in equations (41)–(45) we obtain

$$e^{-\nu} = \operatorname{ch}\beta_1 \tag{46}$$

$$\beta_1'' + \beta_1'^2 \tanh\beta_1 + \frac{2}{r}\beta_1' - \frac{2}{r^2}\tanh\beta_1 = 0. \tag{47}$$

Equation (47) can be integrated twice to give

$$\sinh\beta_1 = \frac{1}{r^2}(Ar^3 + B) \tag{48}$$

where A, B are arbitrary constants. From (46) and (48) we have

$$e^{2\nu} = \left\{1 + \frac{1}{r^4}(Ar^3 + B)^2\right\}^{-1}$$

so that the solution of the field equations is of the following form.

Solution I

$$ds^2 = dt^2 - \left\{1 + \frac{1}{r^4}(Ar^3 + B)^2\right\}^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{49}$$

For the metric (35) and ennuple given by (40), the same components of $C^{\alpha}{}_{\beta\gamma}$ are non-zero as in the previous case. Again by considering the (2, 2) equation we find

$$\beta_2 = \beta_3 = 0$$

and, as a consequence, the only field equations which are not identically zero are

$$C^{\alpha}{}_{11;\alpha} \equiv \mu'^2 = 0 \tag{50}$$

$$C^{\alpha}{}_{22;\alpha} \equiv -a \, e^{-\nu}(\beta_1' \operatorname{sh}\beta_1 + \mu' \operatorname{ch}\beta_1) = 0 \tag{51}$$

$$C^{\alpha}{}_{33;\alpha} \equiv C^{\alpha}{}_{22;\alpha} \sin^2\theta$$

$$C^{\alpha}{}_{14;\alpha} \equiv -(e^{\mu-\nu}\beta_1')' = 0. \tag{52}$$

From these equations we see that $\mu' = 0$, so that we can take $\mu = 0$. Also, $\beta_1' = 0$ and ν is arbitrary. Hence, in this case the solution of the field equations is

$$ds^2 = dt^2 - e^{\nu} \, dr^2 - a^2(d\theta^2 + \sin^2\theta \, d\phi^2).$$

By a simple transformation of the radial coordinate this can be written in the following form.

Solution II

$$ds^2 = dt^2 - dr^2 - a^2(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{53}$$

where r now denotes the new radial coordinate.

It should be noted that in this case β_1 is a constant, which may be zero. In the case of solution I we see that β_1 cannot be a constant.

6. Discussion of the solutions

Consider first solution II. Under the transformation

$$r = \ln \rho^a$$

the metric (53) becomes

$$\mathrm{d}s^2 = \mathrm{d}t \;-\; \frac{a^2}{\rho^2}(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2 + \rho^2 \sin^2\theta\,\mathrm{d}\phi^2)$$

which can be written in the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{a^2}{\rho^2}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{54}$$

introducing Cartesian coordinates in the underlying 3-space. If we take $\beta_1 = 0$, as we may for this solution, then, under the transformation which takes (53) into (54), the ennuple becomes

$$h_i^s = \begin{bmatrix} \frac{\rho}{a} & & & 0 \\ & \frac{\rho}{a} & & \\ & & \frac{\rho}{a} & \\ 0 & & & 1 \end{bmatrix}. \tag{55}$$

This corresponds to the general ennuple $h_i^\alpha = \mathscr{H}\mathscr{F}$ with all six parameters equal to zero, i.e. $\mathscr{H}$ is the unit matrix. Hence, we say that the solution II is *reducible to a zero-parameter solution.* This means that solution II can be obtained by using the simplest possible ennuple, if we use a coordinate system in which the metric can be written in the form

$$ds^2 = e^{2\mu}\,\mathrm{d}t^2 - e^{2\nu}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{56}$$

Turning now to solution I, when $B = 0$ the transformation

$$Ar = \frac{\bar{r}}{1 - \frac{1}{4}\bar{r}^2}$$

changes (49) into the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{R^2}{(1 - \frac{1}{4}r^2)^2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2) \tag{57}$$

where $R = 1/A$ and we have dropped the bar. This is a form of the Robertson–Walker metric, namely

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{\mathscr{R}^2(t)}{(1 + \frac{1}{4}kr^2)^2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2)$$

with $k = -1$ and $\mathscr{R}$ constant. Under the further transformation

$$r = \frac{2}{\rho}\{\tau - (\tau^2 - \rho^2)^{1/2}\} \qquad t = \tfrac{1}{2}R \ln(\tau^2 - \rho^2)$$

the metric (57) becomes

$$\mathrm{d}s^2 = \frac{R^2}{\tau^2 - \rho^2}(\mathrm{d}\tau^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2) \tag{58}$$

where x, y, z are Cartesian coordinates in the underlying 3-space. Under the coordinate transformations which result in the form (58) the parameter β_1 is removed from the ennuple, which then takes the form

$$h_i^{\alpha} = \frac{(\tau^2-\rho^2)^{1/2}}{R}\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

so that this solution is also reducible to a zero-parameter solution although, unlike solution II, the metric tensor is time-dependent in the zero-parameter form.

When $B \neq 0$ there appears to be no coordinate transformation which will remove the β_1 or any of the circular parameters. In this case we say that the solution is an *irreducible four-parameter solution.*

Solution I with $B = 0$ and solution II are both symmetric spaces in the sense of Cartan, i.e. their curvature tensors satisfy

$$R_{\alpha\beta\gamma\delta;\epsilon} = 0.$$

It is noticeable that these two solutions not only have this property in common, but are both reducible to zero-parameter solutions. However, there is no evidence to suggest that there is any connection between the two properties.

The physical interpretation of these solutions presents some difficulties. Kilmister showed that the field equations satisfy Laplace's equation $\nabla^2\phi = 0$ in the weak-field approximation where, as usual, ϕ is given by

$$g_{44} = 1+2\phi.$$

In both solutions found here, $g_{44} = 1$, i.e. $\phi = 0$, so Laplace's equation is satisfied trivially. For weak fields and slowly moving bodies the geodesic equations for both solutions become

$$\frac{d^2r}{dt^2} = 0$$

and so give Newton's equation of motion in the absence of a gravitational field. In the Schwarzschild solution this weak-field approximation is used to identify the constant of integration m, but in the case of solution I the constants A, B cannot be so identified.

When $B = 0$, solution I can be written in the form (57). This form is known to represent a homogeneous, isotropic space–time and so cannot describe the field of an isolated mass particle. It follows that, in the context of Kilmister's vacuum field equations, the solution describes an empty space–time. Like the de Sitter solution of Einstein's field equation this is an empty but curved space–time and so is in disagreement with Mach's principle.

When $B \neq 0$, solution I has an intrinsic singularity at $r = 0$ which cannot be removed by coordinate transformation since the scalar

$$R_{\alpha}{}^{\beta}R^{\alpha}{}_{\beta} = 6\left(A^2 - \frac{AB}{r^3} - \frac{2B^2}{r^6}\right)^2$$

has an infinity at $r = 0$. The geodesics are of the form

$$\frac{d^2u}{d\theta^2} + u = \frac{l^2-1}{h^2}\left(-\frac{A^2}{u^3} + AB + 2B^2u^3\right) - 3ABu^2 - 3B^2u^5$$

where $l = \mathrm{d}t/\mathrm{d}s$, $h = r^2\, \mathrm{d}\phi/\mathrm{d}s$ and $u = 1/r$. This is not approximately an ellipse and so cannot represent the orbit of a test particle about a central massive particle. Also when $A = 0$, the space–time (49) is asymptotically flat, which is one of the properties of the Schwarzschild solution to which Kilmister objected.

Solution II has no singularities and the weak-field approximation suggests that there is no central massive particle present. The metric (53) is not asymptotically flat, the only non-zero component of the curvature tensor being

$$R_{2323} = -\sin^2\theta$$

which is independent of the radial coordinate. The surface of a sphere of radius $r = r_0 =$ constant is $4\pi a^2$, which is independent of r_0. Bonnor (1962), in the course of an investigation into Birkhoff's theorem, rejected a solution with similar properties as having no physical significance. However, Robinson (1959) discovered a solution of the Maxwell–Einstein equations with this same property and this solution has been given a physical interpretation by Dolan (1968), so we shall not reject the solution for the reasons given by Bonnor.

The geodesics of this solution, for motion in the $\theta = \frac{1}{2}\pi$ plane, integrate to give

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \text{constant}$$

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \text{constant}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \text{constant}.$$

Hence, a test particle projected from the origin in a radial direction will continue to travel with uniform velocity as in Minkowski space–time, which justifies our suggestion that this solution does not represent the field of a massive body. In this case space–time is not flat, and, since it is empty, the curvature could be ascribed to the presence of distant matter (the 'fixed stars') which determines the inertial frame in which the particle travels with uniform velocity. This is in accord with Mach's principle.

Unfortunately the non-radial motion of a test particle is somewhat curious for a reasonable physical interpretation. If a test particle is set in motion from a point with a velocity in a direction perpendicular to the radius vector at that point, it will describe a circle with constant angular velocity since $\mathrm{d}\phi/\mathrm{d}t$ is a constant. If it is set in motion with a velocity in any other direction it will spiral away from the origin with constant angular and radial velocities. It is difficult to see how this behaviour of a test particle can be reconciled with the physical picture of a space–time containing no matter other than that at a very great distance from the origin.

Despite its difficulties, solution I is the only solution which it seems possible to associate with the presence of distant matter. It is interesting, but perhaps idle, to speculate whether this physical property can be related to the fact that this solution is, in its static form, the only solution obtainable from the simplest possible zero-parameter ennuple.

7. Order of multiplication of the Lorentz matrices

In our discussion of the ennuple to be used in obtaining the components of the field equations we have taken the order of multiplication of the Lorentz matrices in the ennuple to be that given by (37) and (38), i.e.

$$\mathscr{H} = \Theta H_{34} H_{24} H_{14}.$$

We now consider the effects of taking other orders of multiplication.

For the moment we assume that is it desirable to keep the product Θ of the circular matrices intact in the form (33), since Θ is related to the proper ennuple in Minkowski space–time. The three hyperbolic matrices can be permuted amongst themselves and can collectively or separately pre-multiply or post-multiply Θ. When the order of multiplication is changed, the four ennuple vectors are altered and it does not necessarily follow that the components of the field equations will be identical in the different cases. We have computed the field equations in all the possible cases and have solved them with the following results.

When H_{14} is to the right of Θ, whatever the positions and order of H_{24}, H_{34}, e.g. $\Theta H_{24} H_{14} H_{34}$, $H_{24} \Theta H_{34} H_{14}$, $H_{24} H_{34} \Theta H_{14}$, etc., the only solutions of the resulting field equations are the solutions I and II already found. The values of the parameters β_1, β_2, β_3 are not necessarily the same in each case. For example, when

$$\mathscr{H} = H_{24} H_{34} \Theta H_{14}$$

the solution I has $\beta_2' = \beta_3' = 0$ rather than $\beta_2 = \beta_3 = 0$ and solution II has $\beta_1 = \beta_2' = 0, \beta_3' = kr$, where k is an arbitrary constant, rather than $\beta_1' = \beta_2 = \beta_3 = 0$. Solution II still qualifies as a zero-parameter solution when written in the form (54) because the solution still holds when $\beta_1 = \beta_2 = \beta_3 = 0$, unlike solution I which, in its static form, cannot have $\beta_1 = 0$.

When H_{14} is to the left of Θ, whatever the positions and order of H_{24}, H_{34}, it is found that solution II is, in every case, the only solution of the field equations. The values of the hyperbolic parameters are again different for the various cases, but as before the solution still holds when all three parameters are zero.

Finally, we drop the assumption that the circular matrices must be grouped together in the form Θ and consider the case when the six Lorentz matrices are multiplied together in random order such as

$$\mathscr{H} = H_{24} H_{12} H_{14} H_{31} H_{34} H_{23}.$$

We have not investigated every one of the large number of such possible products, but from those that we have investigated it is clear that whenever such a product is used the hyperbolic matrices that are interposed between the circular matrices are reduced to unit matrices, i.e. the hyperbolic parameters are zero and hence the circular matrices again group together to form the matrix Θ. For example, in the case mentioned above we find $\beta_1 = \beta_3 = 0$, so that H_{14} and H_{34} are unit matrices and hence

$$\mathscr{H} = H_{24} \Theta.$$

In such cases, if H_{14} is to the right of the last circular matrix, then both solutions I and II are found; otherwise only solution II is found.

There are two striking features of these results apart from their remarkable consistency. Firstly, solution II is a solution, and often the only solution, of the field equations for every order of multiplication of our particular forms of the six Lorentz matrices. Secondly, the circular product matrix Θ associated with the proper ennuple in spherical polar coordinates seems to justify the importance that we attach to it by resisting all attempts to 'split' it with hyperbolic matrices. This is very satisfying but unfortunately we do not understand why it possesses this property.

8. Conclusion

The solutions of Kilmister's field equation found here are disappointing in that they do not appear to describe the required physical situation of the gravitational field of a massive body, nor do they appear to be in agreement with Mach's principle as required by Kilmister. This is a condemnation of the particular field equations proposed by Kilmister, but there are many other sets of field equations that can be formed from the ennuple, and one such set may give the desired results.

Kilmister's equations suffer from the disadvantage that they are not derivable from a variational principle. In a subsequent paper we hope to describe field equations which can be derived from a variational principle in which the $h_i{}^{\alpha}$, rather than the $g_{\alpha\beta}$, are the quantities to be varied. Even if the solutions of the other field equations are also not in accord with Mach's principle it would appear that an investigation of this type is necessary, and perhaps fruitful, if we are to follow Plebanski's suggestion that the ennuple represents something physical.

Ennuple systems have been used successfully in a number of investigations, usually to prove some general results. When they are applied to particular space–times, the calculations are bedevilled by the dreadful complications of six parameters, each being functions of all four variables. To be useful the ennuple must be simplified in some way and in this paper we have shown the dangers of too much or incorrect simplification, and have demonstrated how simplification can be carried out using the proper ennuple so that solutions of the field equations can be found with the minimum of effort. We hope that the methods used here will be found to be useful in other calculations involving the use of ennuple vectors in particular space–times.

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References

BONDI, H., PIRANI, F. A. E., and ROBINSON, I., 1959, *Proc. R. Soc.* A., **251,** 519–33.

BONNOR, W. B., 1962, in *Recent Developments in General Relativity* (Oxford: Pergamon Press).

CARTAN, E., 1951, *Leçons sur la Géométrie des Espaces de Riemann,* 2nd edn (Paris: Gauthier-Villars).

DOLAN, P., 1968, *Commun. Math. Phys.,* **9,** 161–8.

EINSTEIN, A., 1928, *Sitz. Preuss. Akad. Wiss.,* **1,** 217.

EISENHART, L. P., 1926, *Riemannian Geometry* (Princeton: Princeton University Press), p. 97.

KILMISTER, C. W., 1967, *Perspectives in Geometry and Relativity,* Ed. B. Hoffmann (Indiana: Indiana University Press).

PLEBANSKI, J., 1962, *Proc. 4th Int. Conf. on Gravitation,* Ed. L. Infeld (Paris: Gauthier-Villars).

ROBINSON, I., 1959, *Bull. Acad. Pol. Sci. Sér. Sci. Math. Astr. Phys.,* **7,** 351–2.

SCHOUTEN, J. A., 1954, *Ricci Calculus,* 2nd edn (Berlin: Springer-Verlag).